

Retarded Correlators of Charge Transport in a Magnetic Field

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Outline

- 1 Motivation & Introduction
- 2 Kinetic Theory in RTA
- 3 Retarded Current-Current Correlators
- 4 Hydrodynamic Limit: Diffusion Mode
- 5 Non-Hydrodynamic Branch Cut
- 6 Weak-Coupling Limit
- 7 Summary and Outlook

Motivation

- To characterize the collective excitations and transport properties of many-body systems, retarded correlators serve as a central tool.
- The retarded correlators are defined by

$$G_{\mu,\nu}^J(t-t', \mathbf{x}-\mathbf{x}') = -i\theta(t-t')\langle [J_\mu(t, \mathbf{x}), J_\nu(t', \mathbf{x}')] \rangle,$$
$$G_{\alpha\beta,\gamma\delta}^T(t-t', \mathbf{x}-\mathbf{x}') = -i\theta(t-t')\langle [T_{\alpha\beta}(t, \mathbf{x}), T_{\gamma\delta}(t', \mathbf{x}')] \rangle,$$

which describe the causal response induced in a system by an external perturbation.

- After analytic continuation into the complex ω -plane:
 - **Hydrodynamic poles:** charge diffusion and sound modes, long-lived collective excitations
 - **Non-hydrodynamic poles:** associated with microscopic fast degrees of freedom and determine the regime of validity of hydrodynamic descriptions
 - **Branch cuts:** a continuum of microscopic wave-particle resonances associated with Landau and cyclotron damping.

From Hydrodynamics to Kinetic Theory

Hydrodynamic regime

Hydrodynamics is an effective theory of many-body systems in the long-wavelength and low-frequency regime,

$$|\mathbf{k}|\ell_{\text{micro}} \ll 1, \quad |\omega|\tau_{\text{micro}} \ll 1.$$

Therefore, retarded correlators calculated within hydrodynamics describe only the infrared response and do not resolve the full microscopic singularity structure.

Our approach

We use relativistic kinetic theory, formulated through the Boltzmann equation, to calculate the retarded correlator beyond the hydrodynamic limit:

$$J^\mu = J_0^\mu - G_{JJ}^{\mu,\nu} \delta A_\nu - \frac{1}{2} G_{JT}^{\mu,\alpha\beta} \delta g_{\alpha\beta} + \dots, \quad T^{\mu\nu} = T_0^{\mu\nu} - \frac{1}{2} G_{TT}^{\mu\nu,\alpha\beta} \delta g_{\alpha\beta} - G_{TJ}^{\mu\nu,\alpha} \delta A_\alpha + \dots$$

The momentum-dependent distribution function allows us to investigate both hydrodynamic poles and non-hydrodynamic structures.

Setup: Kinetic Theory in RTA

- The relativistic Boltzmann equation is taken in the constant-relaxation-time approximation:

$$p^\mu \partial_\mu f + F^{i\beta} p_\beta \frac{\partial}{\partial p^i} f = -\frac{p_\mu u^\mu}{\tau_R} (f - f_{\text{eq}}).$$

- Background:** Uniform magnetic field $\mathbf{B}_0 = B_0 \hat{x}$.
 - Global equilibrium: $f_0(\mathbf{p}) = \exp[(\mu_0 - u_0^\mu p_\mu)/T_0]$.
 - Temperature T_0 , chemical potential μ_0 , flow velocity $u_0^\mu = (1, 0, 0, 0)$.
- Introduce a small external $U(1)$ gauge perturbation $\delta A^\mu(t, \mathbf{x})$:**

$$\begin{aligned}\mu(t, \mathbf{x}) &= \mu_0 + \delta\mu(t, \mathbf{x}), \\ f(t, \mathbf{x}, \mathbf{p}) &= f_0(\mathbf{p}) + \delta f(t, \mathbf{x}, \mathbf{p}), \\ f_{\text{eq}}(t, \mathbf{x}, \mathbf{p}) &= f_0(\mathbf{p}) + \frac{f_0(\mathbf{p})}{T_0} \delta\mu(t, \mathbf{x}).\end{aligned}$$

Analytic Solution of the Linearized Boltzmann Equation

- Small perturbations $\delta f(t, \mathbf{x}, \mathbf{p})$ are required to obey

$$p^\mu \partial_\mu (f_0 + \delta f) + (F_0^{i\beta} + \delta F^{i\beta}) p_\beta \frac{\partial}{\partial p^i} (f_0 + \delta f) = -\frac{p^0}{\tau_R} (\delta f - \delta f_{\text{eq}}).$$

- After linearizing the equation and performing the Fourier transform:

$$\left[\left(\frac{1}{\tau_R} - i\omega + i\mathbf{k} \cdot \mathbf{v} \right) + B_0 \left(v_z \frac{\partial}{\partial p_y} - v_y \frac{\partial}{\partial p_z} \right) \right] \delta \tilde{f}(\omega, \mathbf{k}, \mathbf{p}) = \frac{1}{\tau_R} \delta \tilde{f}_{\text{eq}}(\omega, \mathbf{k}, \mathbf{p}) + \frac{f_0(\mathbf{p})}{T_0} \mathbf{v} \cdot \tilde{\mathbf{E}}(\omega, \mathbf{k}).$$

- Use cylindrical symmetry: $p_y = p_\perp \cos \phi$, $p_z = p_\perp \sin \phi$,

$$v_z \frac{\partial}{\partial p_y} - v_y \frac{\partial}{\partial p_z} = -\frac{1}{p^0} \frac{\partial}{\partial \phi}.$$

- Exact series solution** (Jacobi-Anger expansion):

$$\delta \tilde{f}(\omega, \mathbf{k}, \mathbf{p}) = \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=-1}^1 \frac{\Delta_m J_n(\xi) J_l(\xi)}{\frac{1}{\tau_R} - i(\omega - k_x v_x) + i(n-m)\omega_c} e^{i(l+m-n)\phi} e^{-i(l-n)\psi}.$$

- Here $\xi = k_\perp p_\perp / B_0$, $\omega_c = B_0 / p^0$, $\psi = \arctan(k_z / k_y)$ and $\phi = \arctan(p_z / p_y)$.

Retarded Current–Current Correlators in a Magnetic Field

- Current fluctuation:

$$\delta J^\mu(\omega, \mathbf{k}) = \int \frac{d^3 p}{(2\pi)^3 p^0} p^\mu \delta \tilde{f}(\omega, \mathbf{k}, \mathbf{p}).$$

- Retarded current-current correlators are given by

$$G_{JJ}^{\mu\nu}(\omega, \mathbf{k}) \equiv \frac{\delta J^\mu}{\delta A_\nu} = -\chi \delta_0^\mu \delta_0^\nu + \omega \left[K^{\mu\nu} + \frac{i}{\tau_R \chi} \frac{K^{\mu 0} K^{0\nu}}{1 - \frac{i}{\tau_R \chi} K^{00}} \right],$$

where the kernel is

$$K^{\mu\nu}(\omega, \mathbf{k}) = \sum_{l=-\infty}^{\infty} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{f_0(\mathbf{p})}{T_0} \frac{\mathcal{L}_l^\mu(\mathbf{k}, \mathbf{p}) \mathcal{R}_l^\nu(\mathbf{k}, \mathbf{p})}{\omega - k_x v_x - l\omega_c + i/\tau_R}.$$

- $\mathcal{L}_l^\mu, \mathcal{R}_l^\nu$ involve Bessel functions $J_l(\xi)$.
- **Ward identities** are exactly satisfied: $k_\mu G_{JJ}^{\mu\nu} = 0, k_\nu G_{JJ}^{\mu\nu} = 0$.

Hydrodynamic Limit: Diffusion Mode

$$G_{JJ}^{00}(\omega, \mathbf{k}) = -\chi + \frac{\omega K^{00}(\omega, \mathbf{k})}{1 - \frac{i}{\tau_R \chi} K^{00}(\omega, \mathbf{k})}, \quad K^{\mu\nu}(\omega, \mathbf{k}) = \sum_{l=-\infty}^{\infty} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{f_0(\mathbf{p})}{T_0} \frac{J_l^2(\xi)}{\omega - k_x v_x - l\omega_c + i/\tau_R}.$$

- The poles of the retarded correlators are determined by the pole equation:

$$1 - \sum_l \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{J_l^2(\xi)}{1/\tau_R - i(\omega - k_x v_x) + il\omega_c} \frac{f_0(\mathbf{p})}{T_0 \tau_R \chi} = 0.$$

- Expand in small $\omega, \mathbf{k}$ (hydrodynamic limit):

Diffusion Mode: $\omega(\mathbf{k}) = -iD_L k_x^2 - iD_T k_\perp^2 + \mathcal{O}(k^4),$

$$\frac{D_L}{\tau_R} \equiv \langle v_x^2 \rangle = \frac{1}{3} + \hat{m} \frac{G_{1,3}^{2,1} \left(\frac{\hat{m}^2}{4} \middle| -\frac{1}{2}, \frac{1}{2}, 0 \right)}{12K_2(\hat{m})} + \hat{m} \frac{\pi}{6K_2(\hat{m})}, \quad \frac{D_T}{\tau_R} \equiv \frac{1}{2} \left\langle \frac{v_\perp^2}{1 + \omega_c^2 \tau_R^2} \right\rangle.$$

- Anisotropy:** Longitudinal diffusion unaffected by B_0 ; transverse diffusion suppressed.

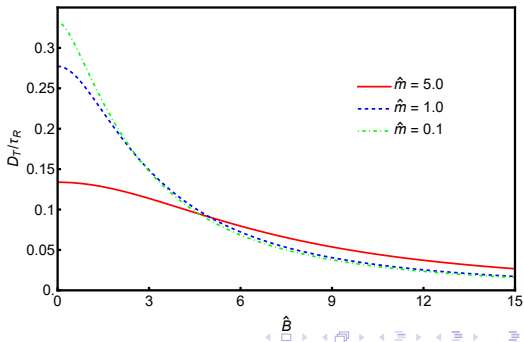
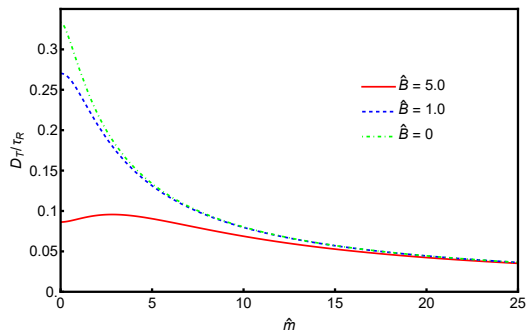
Transverse Diffusion: Mass and Magnetic Field Dependence

- Weak-field expansion: $\frac{D_T}{\tau_R} = \langle v_x^2 \rangle - \frac{\hat{B}^2}{2\hat{m}^2} \left[1 - \hat{m} \frac{\kappa_1(\hat{m})}{\kappa_2(\hat{m})} + (\hat{m}^2 - 3) \frac{D_L}{\tau_R} \right] + \mathcal{O}(\hat{B}^4)$
- Strong-field expansion: $\frac{D_T}{\tau_R} = \frac{\hat{m}}{\hat{B}^2} \frac{\kappa_3(\hat{m})}{\kappa_2(\hat{m})} - \frac{1}{\hat{B}^4} \left[5\hat{m}^2 \frac{\kappa_4(\hat{m})}{\kappa_2(\hat{m})} + \hat{m}^3 \frac{\kappa_3(\hat{m})}{\kappa_2(\hat{m})} \right] + \mathcal{O}\left(\frac{1}{\hat{B}^6}\right)$

Physical picture: Collision-driven 2D random walk with Larmor radius $r_L = p_\perp / B_0$.

$$D_T \sim \langle r_L^2 \rangle / \tau_R = \langle p_\perp^2 \rangle / B_0^2 \tau_R$$

Heavier particles $\Rightarrow$ larger $\langle p_\perp^2 \rangle \Rightarrow$ stronger transverse diffusion .



Non-hydrodynamic Branch Cut

- Non-hydrodynamic contributions appear as branch cuts in the complex ω -plane.
- For the density-density correlator,

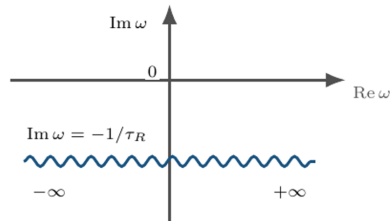
$$G_{JJ}^{00}(\omega, \mathbf{k}) = -\chi + \frac{\omega K^{00}(\omega, \mathbf{k})}{1 - \frac{i}{\tau_R \chi} K^{00}(\omega, \mathbf{k})}.$$

$$K^{00}(\omega, \mathbf{k}) = \sum_{l=-\infty}^{\infty} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{f_0(\mathbf{p})}{T_0} \times \frac{J_l^2(\xi)}{\omega - k_x v_x - l\omega_c + i/\tau_R}.$$

- The kernel K^{00} contains the denominator

$$\omega - k_x v_x - l\omega_c + \frac{i}{\tau_R},$$

which determines the branch cuts.



Origin of the branch cut

For fixed $\mathbf{p}$ and l , the integrand has a simple pole at

$$\omega = k_x v_x + l\omega_c - \frac{i}{\tau_R}.$$

As $\mathbf{p}$ varies continuously, these poles form a continuum. The momentum integration converts this continuum into a branch cut.

The cut lies at $\text{Im } \omega = -1/\tau_R$, with $\text{Re } \omega$ covering the entire real axis

Physical Interpretation of the Branch Cut

What does the branch cut represent physically?

The branch cut represents a continuum of microscopic wave-particle resonances rather than a single discrete collective mode.

Resonance condition

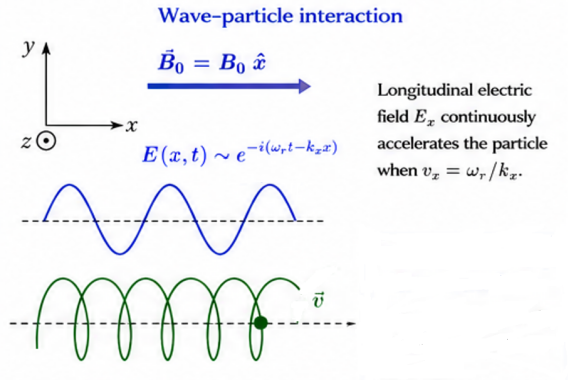
The real part of the singularity condition is

$$\omega_r = k_x v_x + l \omega_c, \quad \omega_r \equiv \text{Re} \omega, \quad \omega_c = \frac{B_0}{p^0}.$$

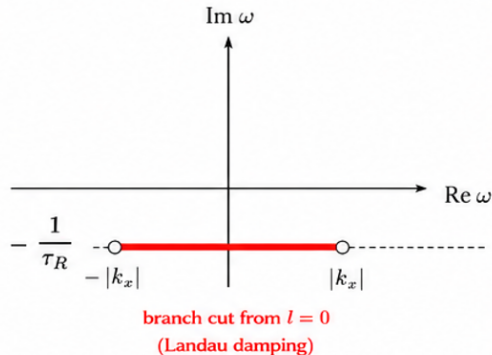
Each integer l labels a distinct wave-particle resonance channel:

- $l = 0$: Landau damping.
- $l \neq 0$: Cyclotron damping.

Landau Damping: $l = 0$, $\omega_r = k_x v_x$

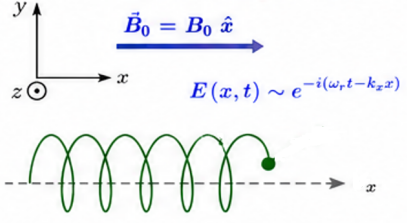
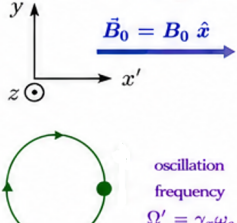


Contribution to the complex ω plane



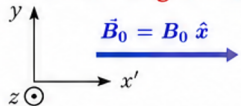
Since $|v_x| < 1$, we have
 $-|k_x| < \omega_r = k_x v_x < |k_x|$.

Cyclotron Damping: $l \neq 0$, $\omega_r = k_x v_x + l\omega_c$

In the laboratory frame		In the longitudinally co-moving frame
 $\vec{B}_0 = B_0 \hat{x}$ $E(x, t) \sim e^{-i(\omega_r t - k_x x)}$ Helical motion in 3D: circular in y-z plane + uniform motion along x with velocity v_x.	Transform to the co-moving frame moving with velocity v_x along $+\hat{x}$	 $\vec{B}_0 = B_0 \hat{x}$ Boost along $+\hat{x}$ with velocity v_x. Particle: no longitudinal motion, pure circular orbit in y-z plane. Wave seen in co-moving frame (Doppler shifted): $\omega'_r = \gamma_x (\omega_r - k_x v_x)$ oscillation frequency $\Omega' = \gamma_x \omega_c$ $\omega_c = \frac{qB_0}{\gamma m}, \quad \gamma_x = \frac{1}{\sqrt{1 - v_x^2}}$ $\Rightarrow \Omega' = \gamma_x \omega_c$

Cyclotron Damping: $l \neq 0$, $\omega_r = k_x v_x + l \omega_c$

In the longitudinally co-moving frame



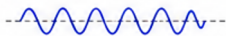
Particle: no longitudinal motion,
pure circular orbit in y - z plane.



oscillation
frequency
 $\Omega' = \gamma_x \omega_c$

Wave seen in co-moving frame
(Doppler shifted):

$$\omega'_r = \gamma_x (\omega_r - k_x v_x)$$



$$\omega_c = \frac{qB_0}{\gamma m}, \quad \gamma_x = \frac{1}{\sqrt{1 - v_x^2}}$$

$$\Rightarrow \Omega' = \gamma_x \omega_c$$



Resonance
in co-moving
frame

Resonance and back to the lab frame

Resonance occurs when the Doppler-shifted
wave frequency matches an integer multiple
of the oscillator frequency:

$$\omega'_r = l \Omega' \quad (l = \pm 1, \pm 2, \dots)$$



Cancel γ_x on both sides

$$\omega_r - k_x v_x = l \omega_c$$

These resonances give rise to cyclotron (harmonic)
damping and produce branch cuts (or poles) at

$$\omega_r = k_x v_x + l \omega_c$$

Weak-coupling Expansion

Expansion in the collision rate

For large τ_R , the retarded density-density correlator admits the expansion

$$G_{JJ}^{00}(\omega, \mathbf{k}) = G_0(\omega, \mathbf{k}) + \frac{1}{\tau_R} G_1(\omega, \mathbf{k}) + \mathcal{O}\left(\frac{1}{\tau_R^2}\right).$$

The leading term is the collisionless Vlasov response:

$$G_0(\omega, \mathbf{k}) = -\chi + \omega \int \frac{d^3p}{(2\pi)^3} \frac{f_0}{T_0} \sum_l \frac{J_l^2(\xi)}{\omega - k_x v_x - l\omega_c + i0^+}.$$

First-order correction:

$$G_1(\omega, \mathbf{k}) = i \left[\frac{\partial G_0}{\partial \omega} + \frac{G_0}{\omega} \left(1 + \frac{G_0}{\chi} \right) \right].$$

- **Exact analytic solution** of linearized Boltzmann equation in a uniform magnetic field using Jacobi-Anger expansion.
- **Full set of retarded current-current correlators** $G_{JJ}^{\mu\nu}(\omega, \mathbf{k})$ obtained, Ward identities verified.
- **Hydrodynamic limit:**
 - Diffusion mode
 - Anisotropic diffusion: D_L unaffected, $D_T \sim 1/B_0^2$ in strong field.
- **Non-hydrodynamic branch cuts:**
 - Located at $\text{Im } \omega = -1/\tau_R$.
 - Physical interpretation: Landau damping ($l = 0$) and cyclotron damping ($l \neq 0$).
- **Weak-coupling limit**

- **Momentum-dependent relaxation time:** More realistic collision kernel.
- **Other external fields:** Vorticity, acceleration.
- **Spin degrees of freedom:** Chiral fermions.

Thank You!

Thank You!

Questions and Comments Welcome.

Based on: Xuan Zhao, Qiuz Sun, Yi Wang, Jin Hu, arXiv:2606.08139